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Technical Analysis based on high and low stock prices forecasts: Evidence for Brazil using a fractionally cointegrated VAR model

Leandro dos Santos Maciel

CFA Society
Brazil

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Abstract

This paper addresses the modeling and forecasting of daily high and low asset prices in the Brazilian stock market using a fractionally cointegrated vector autoregressive model (FCVAR). Forecasts are then used in a simple trading strategy to evaluate the application of technical analysis (TA) for equity shares traded at the BM&FBOVESPA. As a flexible framework, FCVAR is able to account for two fundamental patterns of high and low asset prices: their cointegrating relationship and the long-memory of their difference (i.e., the range), a measure of realized volatility. The analysis comprises the twenty most negotiated stocks at the BM&FBOVESPA during the period from January 2010 to May 2017. Empirical findings indicate a significant cointegration relationship between daily high and low prices, which are integrated of an order close to the unity, as well as the range displays long memory and is in the stationary region in most of the cases. Based on historical data, results support that high and low prices of equity shares are largely predictable and their forecasts can improve TA trading strategies applied on Brazilian stock prices. Further, the fractionally cointegrated approach appears as a potential forecasting tool for market practitioners, improving investment strategies.

Keywords: High and low prices, technical analysis, fractional cointegration, stock market, forecasting.

Resumo

Este artigo considera a modelagem e a previsão dos preços máximo e mínimo diários de ativos financeiros no mercado de ações do Brasil com base em um modelo de vetores autoregressivos fracionalmente cointegrados. As previsões são então utilizadas em uma estratégia de *trading* para se avaliar a aplicação da análise técnica envolvendo ações negociadas na BM&FBOVESPA. O modelo de cointegração fracionada apresenta-se como uma abordagem flexível que considera dois padrões fundamentais da dinâmica de preços máximo e mínimo de ativos financeiros: sua relação de cointegração e a memória longa de sua diferença, i.e. a variação ou *range*, como uma medida de volatilidade realizada. A análise compreende as vinte ações mais negociadas na BM&FBOVESPA no período de janeiro de 2010 a maio de 2017. Os resultados empíricos indicam que os preços máximo e mínimo são significativamente cointegrados com ordem de integração próxima da unidade, e que a variação apresenta memória longa e, na maioria dos casos avaliados, é estacionária. Com base em dados históricos, verificou-se que os preços máximo e mínimo das ações podem ser previstos, e que as previsões são capazes de aprimorar estratégias de *trading* baseadas em análise técnica quando aplicadas ao mercado de ações brasileiro. Além disso, a abordagem de cointegração fracionada mostra-se como uma ferramenta potencial de previsão em aplicações de estratégias de investimentos por agentes de mercado.

Palavras-chave: Preços máximo e mínimo, análise técnica, cointegração fracionada, mercado de ações, previsão.

1. Introduction

Forecasting the future behavior of asset prices based on historical market data has been a popular and important subject for academic research and practitioners. In particular, technical analysts, or chartists, believe that past stock prices and trading volume may show patterns that indicate future trends[1]. Therefore, trading rules that rely on past information can yield higher profits than those that passively track the whole security market (CAPORIN, RANALDO, & MAGISTRIS, 2013). This idea contradicts the weak-form market efficiency, which states that all information from historical data is already incorporated in current prices (FAMA, 1970).

Numerous research papers investigating the forecasting power of different mechanical trading strategies, charts and patterns have been published over the years (SHYNKEVICH, 2016; CHEN, SU & LIN, 2016; ZHU & ZHOU, 2009; SCHULMEISTER, 2009). For instance, Lo, Mamaysky and Wang (2000) indicate that there are methods to systematically extract outperforming technical patterns. Park and Irwin (2009) also state that participants in different financial markets use technical analysis (TA), besides the lack of substantial support by academics. Indeed, Menkhoff (2010), based on the results of a survey of 692 fund managers in five countries including the United States, argues that 87% of fund managers put at least some importance on technical analysis and that TA becomes the most important forecasting tool in

[1] On the other hand, fundamental analysts state that underlying factors that affect a company's actual business and its future prospects are the determinants of a stock's value.

decision making for shorter-term periods. Concerning the Brazilian financial market, Macedo, Godinho and Alves (2017), Sanvicente (2015), Oliveira, Nobre and Zárate (2013) and Lorenzoni et al. (2007) are examples supporting the applicability of technical analysis[2].

The aim of this work is to contribute to previous literature on technical analysis and also to market practitioners by evaluating a trading strategy based on high and low stock prices forecasts using data from equity shares negotiated at the BM&FBOVESPA, the Brazilian stock exchange. Based on an empirical analysis, the research goal is to answer the following questions: i) are high and low prices of equity shares traded at the BM&FBOVESPA predictable?; ii) which approach is appropriate to model these prices?; iii) can high and low prices forecasts produce profitable results using TA trading strategies?

1.1. Motivation and contribution

Traditional econometric time series models are frequently based on opening and/or closing prices of assets, stock indices, and exchange rates (ARROYO, ESPÍNOLA & MATÉ, 2011). This is useful in many cases, but it may be insufficient in situations where several values are observed at each time period (day, hour, minute). For instance, if only the opening (or closing) asset price is measured daily, the resulting time series will hide the intraday variability and important information is missed (DEGIANNAKIS & FLOROS, 2013; HANIFF & POK, 2010).

Besides intraday time series could be modelled and forecasted, they reveal characteristics such as irregular temporal spacing, strong diurnal patterns and complex dependence, which result in obstacles for traditional time series models. Further, the accurate prediction of the whole sequence of intraday prices for one day ahead is almost impossible in practical situations. These limitations can be alleviated if the high (maximum) and the low (minimum) asset prices are measured at each time period (ENGLE & RUSSEL, 2009).

In particular, daily high and low prices provide valuable information regarding the dynamic process of an asset throughout time. These prices can be seen as references values for investors in order to place buy or sell orders, e.g. through candlestick charts, a popular technical indicator (XIONG, LI & BAO, 2017; CHEUNG & CHINN, 2001). He and Wan (2009) also stated that the highs and lows are referred to prices at which the excess of demand changes its direction.

[2] Nazário, Silva, Sobreiro and Kimura (2017) provide a rich and extensive literature review on technical analysis considering stock markets.

Additionally, high and low prices are related with the concept of volatility. Alizadeh, Brandt and Diebold (2002) show that the difference between the highest and lowest (log) prices of an asset over a fixed sample interval, also known as the (log) range, is a highly efficient volatility measure[3]. Brandt and Diebold (2006) and Shu and Zhang (2006) pointed out that the range-based volatility estimator appears robust to microstructure noise such as bid-ask bounce, which overcomes the limitations of traditional volatility models based on closing prices that fall to use the information contents inside the reference period of the prices, resulting in inaccurate forecasts.

In addition, daily highs and lows can be used as stop-loss bandwidths, providing information about liquidity provisioning and the price discovery process. According to Caporin et al. (2013), high (low) prices are more likely to correspond to ask (bid) quotes; thus, transaction costs and other frictions, such as price discreteness, the tick size (i.e., the minimal increments) or stale prices, might represent disturbing factors. Finally, high and low prices are more likely to be affected by unanticipated public announcements or other unexpected shocks. Therefore, aspects such as market resiliency and quality of the market infrastructure can be determinant (CAPORIN ET AL., 2013).

Although many research has been devoted to the analysis of the predictability of daily market closing prices, few studies based on econometric time series models examined the case of high and low prices, as for instance the works of Baruník and Dvořáková (2015), Caporin et al. (2013), Cheung, Cheung and Wan (2010), Cheung, Cheung, He and Wan (2009), He and Hu (2009), and Cheung (2007)[4]. On the other hand, some empirical studies suggested methodologies designed to process high and low asset prices as interval-valued variables in order to account for the interrelations between the prices (XIONG ET AL. 2017; XIONG ET AL., 2015; YANG, HAN & WANG, 2014; RODRIGUES & SALISH, 2015; ARROYO ET AL., 2011; LIMA & CARVALHO). Besides suggesting the advantages of interval-valued models over univariate time series methods, these approaches are not able to model the dynamic of

[3] The literature that considers the high-low range price as a proxy for volatility dates back to the 1980s with the work of Parkinson (1980).

[4] Caporin et al. (2013) argue that the lack of studies regarding daily high and low asset prices is surprising for at least three reasons: i) the long histories of high and low prices data are readily available; ii) many technical analysis strategies use high and low prices to construct resistance and support levels; iii) these prices can measure market liquidity and transaction costs.

the daily range properly, which may compromise forecasting accuracy.

The literature presented substancial evidence of long memory in the volatility process of asset prices, interest rate differentials, inflation rates, forward premiums and exchange rates (YALAMA & CELIK, 2013; GARVEY & GALLAGHER, 2012; KELLARD, DUNIS & SARANTIS, 2010; BREIDT, CRATO & LIMA, 1998; ANDERSEN & BOLLERSLEV, 1997; BAILLIE, 1996), but few of them studied the range volatility dynamics. Particularly, the work of Caporin et al. (2013) provides empirical evidence of long memory in the ranges of all 30 of the components of the Dow Jones Industrial Average (DJIA) index during the period from 2003 to 2010. By using a long memory forecasting framework, a fractional vector autoregressive model with error correction (FVECM), the authors indicate a potential profit performance of technical analysis strategies based on forecasts of high and low prices. More recently, Baruník and Dvořáková (2015) evaluated the cointegration dynamics between daily high and low stock prices and the long memory properties of their linear combination, i.e. the range, of the main world stock market indices during the 2003-2012 period. The findings suggested that the ranges of all of the indices display long memory and are mostly in the non-stationary region, supporting the recent evidence that volatility might not be a stationary process.

In this context, to answer the question of whether the high and low prices of the equity shares traded at the BM&FBOVESPA are predictable, this work provides an empirical study on the modeling and predictability of these prices by analyzing the time-series properties of daily high and low prices from the twenty of the most widely traded stocks at the BM&FBOVESPA over the period from January 2010 to May 2017. Further, it is suggested a fractionally cointegrated vector autoregressive model (FCVAR), formalized by Johansen (2008) and Johansen and Nielsen (2010, 2012), to model the relationship between highs and lows, as a response the second question proposed in this research. The motivation of this approach is twofold. First, FCVAR modeling is able to capture the cointegrating relationship between high and low prices, i.e. in the short-term they may diverge, but in the long-term they have an embedded convergence path. Second, the range (the difference between high and low prices), as an efficient volatility measure, is assumed to display a long memory, which allows for greater flexibility[5].

[5] The literature considers asset prices to be integrated of order 1, i.e. $I(1)$. However, the choice between stationary, $I(0)$, and non-stationary, $I(1)$, processes can be too restrictive for the degree of integration of daily high

As stated by Baruník and Dvořáková (2015), a more general fractional or long-memory framework, where the series are assumed to be integrated of order d and cointegrated of order less than d, i.e. $CI(d-b)$, where $d, b \in \Re$ and, $0 < b \leq d$, is more useful in capturing the empirical properties of data, in accordance on the evidence of the presence of long memory in the volatility of asset prices[6]. Therefore, the FCVAR framework has the advantage of modeling both the cointegration between highs and lows, and the long-memory property of the range. Finally, concerning the third research question, i.e. if high and low prices forecasts can produce profitable results using TA trading strategies, it is suggested a simple trading strategy based on daily high and low FCVAR forecasts. The results are then compared against traditional benchmarks over different prediction horizons.

The contributions of this work to previous literature can be summarized as follows. First, it provides new empirical evidence of the modeling and predictability of daily high and low prices concerning the Brazilian stock market through a fractional cointegration framework. Second, the paper also addresses an analysis regarding the long memory properties of the range in this economy. Third, the work differs from past literature by analyzing the predictability of high and low prices against traditional time series methods across different prediction horizons (multi-step-ahead forecasts) instead of only one-step-ahed forecasts. In addition, the modeling and forecasting of daily high and low prices have drawn very limited attention in the extant literature, thus, this research contributes in this field, and also by considering market data from an emergent economy like Brazil. Finally, as a practical contribution, the research aims to provide an alternative tool for market practitioners to improve the operations of TA strategies in the Brazilian stock market based on high and low prices.

After this introduction, this paper proceeds as follows. Section 2 describes the data and provides a preliminary analysis of daily high and low prices and the range, focusing on their integration, cointegration, and long memory properties. An empirical fractionally cointegrated model for high and low prices is presented in Section 3. The predictability analysis and the respective results from a TA trading strategy based on high and low prices forecasts are discussed

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and low prices (BARUNÍK & DVOŘÁKOVÁ, 2015). Since these prices can be considered as a possibly fractionally cointegrated relationship, it improves flexibility, mainly when the error correction term from the cointegrating relationship between high and low prices is the range (CHEUNG, 2007; FIESS & MACDONALD, 2002).

[6] A review of the literature of the long memory properties of volatility can be found in Yalama and Celik (2013).

in Section 4. Finally, Section 5 concludes de paper and suggests topics for future research.

2. Dynamic properties of daily high and low prices

This section describes the database and provides an analysis regarding the integration, cointegration and long memory properties of daily high and low stock prices and their diference, the range. Further, tests for the possible fractional cointegration relationship between highs and lows are also presented.

2.1 Database

The dynamic properties and the predictability of daily high and low prices are investigated considering the twenty most traded stocks in the Brazilian stock exchange, the BM&FBOVESPA, for the period from January 4, 2010 to May 31, 2017. Table 1 describes the stocks, their respective companies and industries. For companies with both high liquidity preferred and common shares, only the most traded stock was selected to provide a more representative sample of the Brazilian stock market. The database are comprised by the time series of daily high and low prices within a total of 1,803 observations[7]. For the stocks from Ultrapar Participações SA (UGPA3) and Kroton Educacional SA (KROT3), the samples start in January 2, 2012 (1,253 observations) and January 2, 2013 (1,013 observations), respectively, period in which the liquidity of these assets became more significant.

We consider the daily high log-price, $p_t^H = \log(P_t^H)$, the daily low log-price, $p_t^L = \log(P_t^L)$, and the daily range $R_t = p_t^H - p_t^L$, where P_t^H and P_t^L are the high and low prices at t, respectively. Figure 1 shows the temporal evolution of daily high and low prices of Itaú Unibanco SA (ITUB4) and their diference, i.e. the range. For log-price time series, to improve visibility the daily lows log-prices in Figure 1 are the actual daily low log-prices minus 0.15. Daily highs and lows dynamic suggests the presence of a common trend, indicating that the series are non-stationary and cointegrated. It is worth to note that higher values of the range are associated with the periods of high prices variability, confirming its property as a volatility measure[8].

[7] Data were collected from the Yahoo Finance website (http://finance.yahoo.com/). The respective opening and closing prices were also collected in order to evaluate the results in terms of trading strategies.

[8] The remaining stocks evaluated in this work provide similar patterns regarding the evolution of daily high and low prices and range.

Table 1. Database description of companies shares comprised by the twenty most traded stocks at the BM&FBOVESPA, ordered in terms of liquidity on May, 2017.

Ticker	Company	Industry	Share type
ITUB4	Itaú Unibanco	Banks	Preferred
BBDC4	Banco Bradesco SA	Banks	Preferred
ABEV3	Ambev SA	Food, beverage and tobacco	Common
PETR4	Petroleo Brasileiro SA	Energy	Preferred
VALE5	Vale SA	Materials	Preferred
BBAS3	Banco do Brasil SA	Banks	Common
BRFS3	BRF SA	Food, beverage and tobacco	Common
UGPA3	Ultrapar Participações SA	Energy	Common
CIEL3	Cielo SA	Software and services	Common
KROT3	Kroton Educacional SA	Consumer services	Common
VIVT4	Telefônica Brasil SA	Telecommunication services	Preferred
LREN3	Lojas Renner SA	Retailing	Common
CCRO3	CCR SA	Transportation	Common
RADL3	Raia Drogasil SA	Food and staples retailing	Common
JBSS3	JBS SA	Food, beverage and tobacco	Common
CPFE3	CPFL Energia SA	Utilities	Common
HYPE3	Hypermarcas SA	Pharmaceuticals	Common
EMBR3	Embraer SA	Capital goods	Common
WEGE3	WEG SA	Capital goods	Common
PCAR4	Cia Brasileira de Distribuição	Food and staples retailing	Preferred

2.2 Cointegration and memory properties of highs and lows

To analyze the properties of the daily high and low log-prices and the range, we first evaluate the stationarity of the series. Table 2 provides the Augmented Dickey-Fuller (ADF) (DICKEY & FULLER, 1979) test results for the daily high and low log-prices (p_t^H and p_t^L) as well as the range (R_t), revealing expected findings. Daily high and low prices are unit root processes, i.e. they are non-stationary, under a 0.05 significance level. The daily range is a stationary process, which indicates that daily high and low prices may be cointegrated. Despite these results, it is worth to mention that the ADF test is designed to evaluate the null hypothesis of a unit root against the $I(0)$ alternative, i.e. it has very low power against fractional processes.

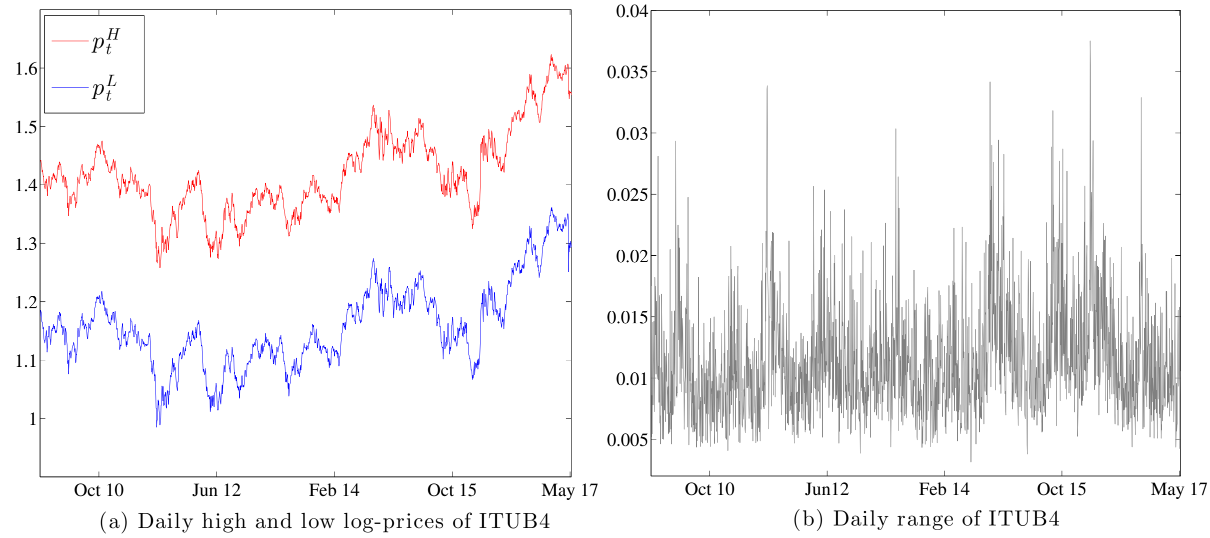


Figure 1. High and low log-prices of ITUB4 (a) and its range (b).

In addition to the ADF test, we performed the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test of Kwiatkowski et al. (1992), appropriate in situations when the tested series are close to being a unit root. The KPSS test results, reported in Table 3, confirm the non-stationarity of the high and low log-prices. However, regarding the range, the results from the KPSS test indicates the presence of a unit root, while the ADF test suggests that the range is stationary. This conflicting results may be caused by the possible long memory property of the range. The results from Table 3 present the KPSS test p-values concerning short lags and long lags in the model. Notice that the results for high and low log-prices for both short and long lags confirm the non-stationarity of the series. On the other hand, when long lags are concerned, the KPSS test results suggest that the range is stationary at a 0.05 significance level (except for PETR4, CCRO3 and HYPE3 stocks, which the range is stationary at a 0.01 significance level). This finding provides evidence on the long memory of the range.

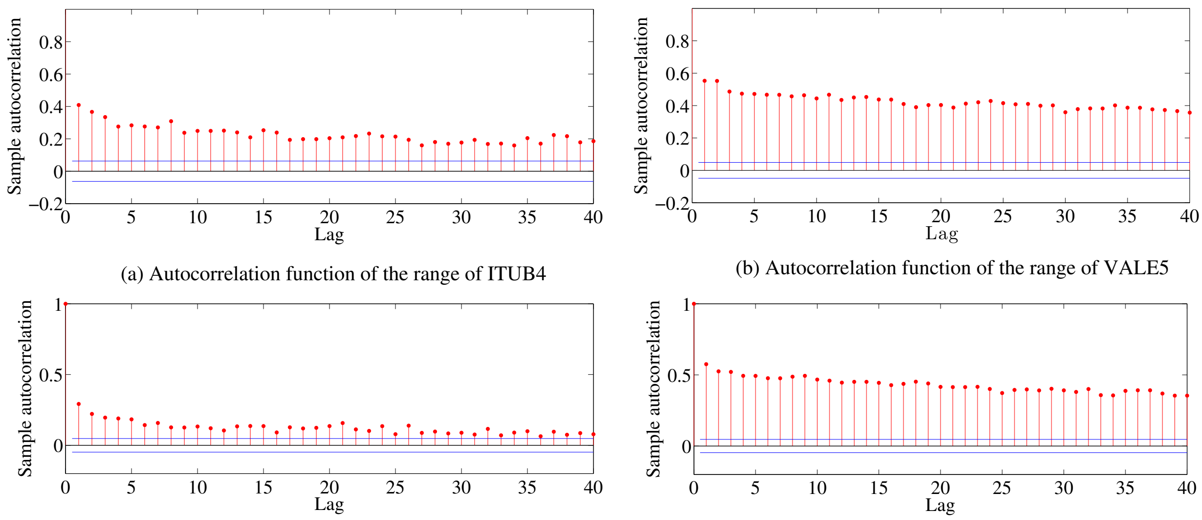
Table 2. P-values of ADF test for unit root for high (H) and low (L) log-prices and range (R) based on levels and first-differences, where c denotes the inclusion of a constant only, t the additional inclusion of a trend for daily high and low log-prices in levels only, and *lags* the number of lags included in the model, selected using the Bayesian Information Criteria (BIC) (SCHWARZ, 1978). The p-value of 0.001 is the minimum reported p-value.

Stock	Model	Lags	ADF_H		ADF_L		ADF_R
			Level	First-differences	Level	First-differences	Level
ITUB4	c	2	0.2816	0.001	0.2034	0.001	0.001
BBDC4	c	2	0.1401	0.001	0.0863	0.001	0.001
ABEV3	c,t	1	0.3836	0.001	0.4005	0.001	0.001
PETR4	c,t	1	0.5726	0.001	0.5669	0.001	0.001
VALE5	c	2	0.3249	0.001	0.3261	0.001	0.001
BBAS3	c,t	2	0.1347	0.001	0.0997	0.001	0.001
BRFS3	c	1	0.3822	0.001	0.3824	0.001	0.001
UGPA3	c	1	0.1844	0.001	0.1514	0.001	0.001
CIEL3	c,t	2	0.6447	0.001	0.6391	0.001	0.001
KROT3	c,t	1	0.4336	0.001	0.4126	0.001	0.001
VIVT4	c,t	2	0.0958	0.001	0.0832	0.001	0.001
LREN3	c	3	0.1612	0.001	0.0959	0.001	0.001
CCRO3	c,t	1	0.4392	0.001	0.3973	0.001	0.001
RADL3	c,t	1	0.7935	0.001	0.8224	0.001	0.001
JBSS3	c	3	0.3501	0.001	0.2272	0.001	0.001
CPFE3	c	1	0.5768	0.001	0.3380	0.001	0.001
HYPE3	c,t	2	0.6420	0.001	0.3111	0.001	0.001
EMBR3	c	1	0.3485	0.001	0.2034	0.001	0.001
WEGE3	c,t	1	0.1085	0.001	0.0923	0.001	0.001
PCAR4	c,t	5	0.5265	0.001	0.7832	0.001	0.001

Table 3. P-values of KPSS test for unit root for high (H) and low (L) log-prices and range (R) based on levels and two lag specifications, short lag and long lag, where c denotes the inclusion of a constant only, t the additional inclusion of a trend for daily high and low log-prices in levels only. Results in bold indicate that series are stationary at a 0.05 significance level. The p-value of 0.01 is the minimum reported p-value.

Stock	Model	$KPSS_H$		$KPSS_L$		$KPSS_R$	
		Short lag	Long lag	Short lag	Long lag	Short lag	Long lag
ITUB4	c	0.01	0.01	0.01	0.01	0.01	**0.0754**
BBDC4	c	0.01	0.01	0.01	0.01	0.01	**0.0937**
ABEV3	c,t	0.01	0.01	0.01	0.01	0.01	**0.0966**
PETR4	c,t	0.01	0.01	0.01	0.01	0.01	0.0478
VALE5	c	0.01	0.01	0.01	0.01	0.01	**0.0858**
BBAS3	c,t	0.01	0.01	0.01	0.01	0.01	**0.0675**
BRFS3	c	0.01	0.01	0.01	0.01	0.01	**0.0583**
UGPA3	c	0.01	0.01	0.01	0.01	0.01	**0.0881**
CIEL3	c,t	0.01	0.01	0.01	0.01	0.01	**0.0689**
KROT3	c,t	0.01	0.01	0.01	0.01	0.01	**0.0656**
VIVT4	c,t	0.01	0.01	0.01	0.01	0.01	**0.0552**
LREN3	c	0.01	0.01	0.01	0.01	0.01	**0.0555**
CCRO3	c,t	0.01	0.01	0.01	0.01	0.01	0.0232
RADL3	c,t	0.01	0.01	0.01	0.01	0.01	**0.0940**
JBSS3	c	0.01	0.01	0.01	0.01	0.01	**0.0778**
CPFE3	c	0.01	0.01	0.01	0.01	0.01	**0.0830**
HYPE3	c,t	0.01	0.01	0.01	0.01	0.01	0.01
EMBR3	c	0.01	0.01	0.01	0.01	0.01	**0.0935**
WEGE3	c,t	0.01	0.01	0.01	0.01	0.01	**0.0999**
PCAR4	c,t	0.01	0.01	0.01	0.01	0.01	**0.0839**

Figure 2 shows the autocorrelation function (ACF) of the ranges of four stocks evaluated in this paper as an example. A high degree of persistence is verified in all cases, with significance autocorrelations even after 30 lags, confirming the results of the KPSS test and the evidence of long memory of the stock price ranges.



(a) Autocorrelation function of the range of ITUB4

(b) Autocorrelation function of the range of VALE5

(c) Autocorrelation function of the range of VIVT4

(d) Autocorrelation function of the range of CPFE3

Figure 2. ACF of daily range of ITUB4 (a), VALE5 (b), VIVT4 (c) and CPFE3 (d).

Similar results on the unit root processes of daily high and low asset prices and the stationarity of the range were also found by Cheung (2007). Therefore, the author suggested a Vector Error Correction Model (VECM) for high and low log-prices. However, due to the high degree of persistence of the range, traditional cointegration analysis may not be satisfactory in explaining the relationship between high and low prices, as already verified by Baruník and Dvořáková (2015) and Caporin et al. (2013), giving rise to the use of the fractionally cointegration framework.

2.3 Testing the fractional cointegration order of high and low prices

The modeling of daily high and low prices as a cointegrated relationship has a particular feature: the "error correction" term, the range, may contain long memory. Differently from Cheung (2007) that used a VECM modeling approach, Baruník and Dvořáková (2015) and Caporin (2013) proposed a fractionally cointegrated model to capture this feature. The previous results reported in this paper, considering the dataset from the Brazilian stock market, also confirm the use of the fractional cointegration framework.

Let $X_t \equiv (p_t^H, p_t^L)'$ be a vector composed by the high and low stock prices, p_t^H and p_t^L, respectively. If the elements of X_t are $I(1)$ and exists a linear combination $\beta' X_t$ that is an $I(0)$ process, X_t is said a cointegrated vector. Robinson and Yajima (2002) indicated that besides the existence of a stable relationship between non-stationary series X_t, i.e. in the short-term they may diverge, but in the long-term they have an embedded convergence path, it does not depend on whether the series are $I(1)$. Therefore, to relax the restriction on the choice between stationary $I(0)$ and non-stationary $I(1)$ processes, the series can be considered an $I(d)$ process with $d \in \Re$, where d is the fractional differencing parameter, fractional degree of persistence or fractional order of integration.

The series X_t is an $I(d)$ process if $u_t = (1-L)^d X_t$ is $I(0)$, with L standing for the lag operator and $d < 0.5$ (ROBINSON & YAJIMA, 2002). If $d \geq 0.5$, X_t is defined as a non-stationary $I(d)$ series with $X_t = (1-L)^{-d} u_t I\{t \geq 1\}$, where $t = 0, \pm 1, \pm 2, \ldots$, and $I\{\cdot\}$ is an indicator function. For $d > 0$ ($d < 0$) the process has long-memory (anti-persistence). If $d = 0$, the process collapses to the random walk, i.e. a stationary process.

To test the fractional order of integration of high and low log-prices and the range of the stocks traded at the BM&FBOVESPA, we employed the univariate exact local Whittle (ELW) estimator, as a semi-parametric approach, proposed by Nielsen and Shimotsu (2007). The method is consistent in the presence or absence of cointegration, and also to both stationary and non-stationary cases. The univariate local exact Whittle estimators for the highs, lows and the range ($\hat{d}^H$, $\hat{d}^L$ and $\hat{d}^R$, respectively) are found by minimizing the following contrast function:

$$Q_{m_d}(d^i, G_{ii}) = \frac{1}{m_d} \sum_{j=1}^{m_d} \left[\log \left(G_{ii} \lambda_j^{-2d^i} \right) + \frac{1}{G_{ii}} I_j \right],\ i = H, L, R, \tag{1}$$

which is concentrated with respect to the diagonal element of the 2×2 matrix G, a finite and nonzero matrix with strictly positive diagonal elements. Under the hypothesis that the spectral density of $U_t = [\Delta^{d^H} p_t^H, \Delta^{d^L} p_t^L, \Delta^{d^R} R_t]$, G satisfies:

$$f_U(\lambda) \sim G \ \text{ as } \ \lambda \to 0, \tag{2}$$

where $f_U(\lambda)$ is the spectral density matrix, I_j the coperiodogram at the Fourier frequency $\lambda_j = \frac{2\pi j}{T}$ of the fractionally differenced series U_t, m_d is the number of frequencies used in the estimation, and T is the sample size (CAPORIN ET AL., 2013). The matrix G is estimated as:

$$\hat{G} = \frac{1}{m_d} \sum_{j=1}^{m_d} Re(I_j), \tag{3}$$

with $Re(I_j)$ standing for the real part of the coperiodogram.

The estimates of the fractional integration order do not imply the presence or absence of cointegration. To test the equality of integration orders, $H_0: \ d^H = d^L = d$, we also employed the test suggested by Nielsen and Shimotsu (2007), which is robust to the presence of fractional cointegration. In the bivariate case under study, the test statistic is:

$$\hat{T}_0 = m_d (S\hat{d})' \left(S \frac{1}{4} \hat{D}^{-1} \left(\hat{G} \odot \hat{G} \right) \hat{D}^{-1} S' + h(T)^2 \right)^{-1} \left(S\hat{d} \right), \tag{4}$$

where $\odot$ is the Hadamard product, $\hat{d} = [\hat{d}^H, \hat{d}^L]$, $S = [1, -1]'$, $h(T) = \log(T)^{-k}$ for $k > 0$, and $D = \text{diag}(G_{11}, G_{22})$.

According to Nielsen and Shimotsu (2007), if the variables are not cointegrated, i.e. the cointegration rank is $r = 0$, $\hat{T}_0 \to \chi^2_1$, while if $r \geq 1$, the variables are cointegrated and $T_0 \to 0$. For significant large values of the test statistic $\hat{T}_0$ with respect to the null density χ^2_1, it evidences against the null hypothesis of the equality of integration orders[9].

The first six columns of Table 4 display the ELW estimates of $\hat{d}^H$, $\hat{d}^L$ and $\hat{d}^R$ for all the stocks under analysis, where the exponent denotes daily high (H), daily low (L) and daily range (R). The estimates of integration orders were calculated base on two specifications of bandwidth, $m_d = T^{0.5}$ and $m_d = T^{0.6}$, as in the works of Nielsen and Shimotsu (2007), Caporin et al. (2013), and Baruník and Dvořáková (2015). For both bandwidths, the order of integration of daily highs and lows are generally high and close to 1, indicating that the series are not stationary. In few cases the unitary integration is exceeded, but not substantially. The difference between high and low prices (the range) is mostly non-stationary ($d > 0$) and displays long memory with parameter $\hat{d}^R < 0.5$, in accordance with the previous findings from the ACF of the ranges (Figure 2). When $m_d = T^{0.5}$ ($m_d = T^{0.6}$), it is observed two (four) cases when the ranges show long memory with parameter $\hat{d}^R$ greater than 0.5, i.e. for VALE5 and CPFE3 (VALE5, PETR4, CPFE3 and WEGE3) stocks[10]. Concerning the bandwidth parameter, the results are not significantly sensitive. Summarizing, the daily high and low prices are not stationary and the range displays long memory, in line with the results of Caporin et al. (2013) and Baruník

[9] For more details refer to Nielsen and Shimotsu (2007).

[10] The empirical findings of Caporin et al. (2013) and Baruník and Dvořáková (2015), using data from developed economies, also suggest the long memory of the range, but with parameter $\hat{d}^R > 0.5$ in most of the cases.

and Dvořáková (2015).

Table 4. Estimates of the fractional order of integration parameter d of high ($\hat{d}^H$) and low ($\hat{d}^L$) log-prices and the range ($\hat{d}^R$) using the exact local Whittle (ELW) estimator, and test statistics for the equality of integration orders ($\hat{T}_0$). All estimates use both $m_d = T^{0.5}$ and $m_d = T^{0.6}$ as bandwidths.

Stock	$\text{ELW}_{m_d=T^{0.5}}$			$\text{ELW}_{m_d=T^{0.6}}$			$\hat{T}_0$	
	$\hat{d}^H$	$\hat{d}^L$	$\hat{d}^R$	$\hat{d}^H$	$\hat{d}^L$	$\hat{d}^R$	$m_d = T^{0.5}$	$m_d = T^{0.6}$
ITAUB4	0.9458	0.9433	0.4016	0.8636	0.8615	0.4904	0.02356	0.00962
BBDC4	0.9681	0.9588	0.4191	0.9173	0.9146	0.4980	0.33188	0.01535
ABEV3	0.9493	0.9565	0.2364	1.0468	1.0342	0.3074	0.16990	0.29797
PETR4	0.9717	0.9717	0.4100	0.9977	1.0083	0.6039	0.11729	0.22006
VALE5	1.0755	1.0718	0.5784	1.1218	1.1320	0.6556	0.05499	0.20446
BBAS3	0.9322	0.9401	0.3795	0.9621	0.9784	0.3852	0.22532	0.53789
BRFS3	1.0009	0.9798	0.2525	1.0301	1.0141	0.3411	1.60855	0.51759
UGPA3	0.9368	0.9388	0.3184	1.0679	1.0556	0.4059	0.01030	0.22491
CIEL3	0.9684	0.9646	0.3502	1.0337	1.0316	0.3671	0.04779	0.00881
KROT3	0.9245	0.9247	0.4344	1.0351	1.0196	0.4796	0.00005	0.29418
VIVT4	0.9003	0.9007	0.3104	0.9167	0.9197	0.4113	0.00049	0.01874
LREN3	0.9582	0.9687	0.2950	0.9899	0.9997	0.3828	0.32599	0.17765
CCRO3	0.8845	0.8986	0.3963	0.9249	0.9257	0.4227	0.62148	0.00118
RADL3	1.0471	1.0449	0.4147	1.0614	1.0690	0.3182	0.01513	0.10242
JBSS3	0.9470	0.9441	0.3211	0.9072	0.8907	0.2396	0.02506	0.45505
CPFE3	0.9697	0.9783	0.6053	0.9893	0.9941	0.7145	0.26326	0.04696
HYPE3	1.0162	0.9940	0.4040	0.9709	0.9475	0.4140	1.77430	1.06194
EMBR3	0.9792	0.9642	0.2566	0.9998	1.0105	0.3395	0.73207	0.22124
WEGE3	1.0402	1.0290	0.4209	0.8242	0.8225	0.5901	0.41793	0.00546
PCAR4	0.9335	1.0340	0.0854	1.0973	1.1035	0.1664	0.80985	0.03506

Regarding the test for the equality of integration orders, the last two columns of Table 4 presents the test statistics estimated with $m_d = T^{0.5}$ and $m_d = T^{0.6}$ as bandwidth parameters. Since the critical value of χ^2_1 is 2.71 in a 90% confidence interval, the null hypothesis of equality of the integration orders cannot be rejected for all tested series (the maximum test statistic is 1.7743), for both bandwidth parameters. The results suggest that a FCVAR modeling approach with the same degree of integration orders $d^H = d^L$ is appropriate for estimating the relationship between the daily high and low prices under study. Notice that the generalization to the presence of fractional cointegration between highs and lows is novel for the modeling of the stocks traded at the Brazilian stock market.

3. FCVAR modeling approach for daily high and low prices

The fractionally cointegrated vector autoregression (FCVAR), formalized by Johansen (2008) and Johansen and Nielsen (2010, 2012), generalize the classical cointegration analysis by allowing X_t to be fractional of order d and cofractional of order $d-b$, which conducts that $\beta' X_t$ should be fractional of order $d-b \geq 0$. This framework allows for the existence of a common stochastic trend, integrated with order d, and the short-term divergences from the long-run equilibrium integrated of order $d-b$. The parameter b is the strength of the cointegrating relationships, called as the cointegration gap (a higher b means less persistence in the cointegrating relationships).

In the FCVAR modeling approach, the usual lag operator and the difference operator are replaced by the fractional lag operator and the fractional difference operator, $L_b = 1 - \Delta^b$ and $\Delta^b = (1-L)^b$, respectively (JOHANSEN & NIELSEN, 2012; NIELSEN & MORIN, 2016). The fractional difference operator is defined by the binomial expansion $\Delta^b Z_t = \sum_{n=1}^{\infty} (-1)^n \binom{b}{n} Z_{t-n}$ (BARUNÍK & DVOŘÁKOVÁ, 2015). Thus, the model is applied to $Z_t = \Delta^{d-b} X_t$. A fractionally cointegrated vector autoregressive FCVAR$_{d,b}(p)$ model for $X_t \equiv (p_t^H, p_t^L)'$ as the vector of high and low prices is described as:

$$\Delta^d X_t = \Delta^{d-b} L_b \alpha \beta' X_t + \sum_{i=1}^{p} \Gamma_i \Delta^d L_b^i X_t + \varepsilon, \quad t = 1, \ldots, T, \tag{5}$$

where α and β are $2 \times r$ matrices comprised by the long-run parameters, $0 \leq r \leq 2$, the rank r is termed the cointegration, or cofractional, rank, $d \geq b > 0$, $\Gamma = (\Gamma_1, \ldots, \Gamma_p)$ are the autoregressive augmentation parameters related to the short-run dynamics, and ε_t is a p-dimensional $i.i.d$ $(0, \Omega)$, with positive-definite variance matrix Ω.

The columns of β constitute the r cointegration (cofractional) vectors such that $\beta' X_t$ are the cointegrating combinations of the variables in the system, i.e. the long-run equilibrium relations. The parameters in α are the adjustment or loading coefficients which represent the speed of adjustment towards equilibrium for each of the variables (NIELSEN & MORIN, 2016). If $d-b<0.5$, $\beta' X_t$ is asymptotically a zero-mean stationary process. Denoting $\Pi = \alpha\beta'$, where the $2\times r$ matrices α and β with $r \leq 2$ are assumed to have full column rank r, the columns of β are then the r cointegrating (cofractional) relationship determining the long-run equilibrium.

Non-zero mean data, $Y_t = \mu + X_t$ for example, can be considered as $\Delta^a Y_t = \Delta^a(\mu + X_t) = \Delta^a X_t$, since $\Delta^a 1 = 0$ for $a > 0$. Thus, this means that the model with $d > b$ is invariant to the inclusion of a restricted constant term ρ. As in Baruník and Dvořáková (2015), the inclusion of a constant term is considered only in the model with $d = b$, which replaces the formulation in (5) by:

$$\Delta^d X_t = L_d \alpha(\beta' X_t + \rho') + \sum_{i=1}^{p} \Gamma_i \Delta^d L_b^i X_t + \varepsilon, \;\; t = 1, \ldots, T, \tag{6}$$

where ρ is the restricted constant term $\mu = \alpha\rho'$, interpreted as the mean level of the long-run equilibrium.

The model parameters are estimated by maximum likelihood as described in Nilsen and Morin (2016). Before estimating the FCVAR models for daily high and low prices of the stocks traded at BM&FBOVESPA, it is required the use of an appropriate approach to test and determine the cointegration rank in the model, described as follows.

3.1 Cointegration rank in the presence of long memory

Cointegration rank testing in the presence of long memory differs from traditional tests for integration (JOHANSEN, 1991). A time series X_t is fractionally cointegrated $CI(d,b)$ if X_t has $I(d)$ elements and for some $b > 0$, exists a vector β such that $\beta' X_t$ is integrated of order $(d-b)$. We first apply the cointegration rank test proposed by Nielsen and Shimotsu (2007), that allows for both stationary and non-stationary fractionally integrated processes. The test is based on the exact local Whittle estimate of d, used to examine the rank of the spectral density matrix G and its eigenvalues. In the bivariate case under study, the test estimates the rank r by:

$$\hat{r} = \arg \min_{u=0,1} L(u), \tag{7}$$

where

$$L(u) = v(T)(2-u) - \sum_{i=1}^{2-u} \hat{\delta}_i, \tag{8}$$

for some $v(T) > 0$ which satisfies

$$v(T) + \frac{1}{m_L^{1/2} v(T)} \to 0, \tag{9}$$

with $\hat{\delta}_i$ as the i-th eigenvalue of $\hat{G}$, and m_L a new bandwidth parameter.

The estimation of matrix G involves two steps. First, $\hat{d}^H$ and $\hat{d}^L$ are obtained first using (1) with m_d as bandwidth parameter. Given $\bar{d}_* = (\hat{d}^H + \hat{d}^L)/2$, the matrix G is estimated as follows:

$$\hat{G} = \frac{1}{m_L} \sum_{j=1}^{m_L} Re(I_j), \tag{10}$$

such that $m_L/m_d \to 0$. The estimates of G are robust to all different choices of m_d and m_L (NIELSEN & SHIMOTSU, 2007).

Table 5 displays the results of the cointegration rank test of Nielsen and SHimotsu (2007) using $m_d = T^{0.6}$ and $m_L = T^{0.5}$ for both cases where $v(T) = m_L^{-0.45}$ and $v(T) = m_L^{-0.05}$. The results suggest that there is one cointegration relationship. In all cases $L(1) < L(0)$ and this can be taken as strong evidence in favor of fractional cointegration between p_t^H and p_t^L so that the expression in (7) is minimized in correspondence of $r = 1$.

In addition, the cointegration rank test proposed by Johansen and Nielsen (2012) was also considered. In the FCVAR framework, the hypothesis $H_r : \text{rank}(\Pi) = r$ is tested against the alternative $H_n : \text{rank}(\Pi) = n$. Let $L(d,b,r)$ be the profile likelihood function given rank r, where (α, β, Γ) have been concentrated out by regression and reduced rank regression (NIELSEN & MORIN, 2016). For the model with a constant, the test concerns the hypothesis $H_r : \text{rank}(\Pi, \mu) = r$ against $H_n : \text{rank}(\Pi, \mu) = n$, with $L(d,r)$ as profile likelihood function given rank r, where the parameters $(\alpha, \beta, \rho, \Gamma)$ have been concentrated out by regression and reduced rank regression.

Table 5. Estimates of the fractional cointegration rank test statistics and their respective eigenvalues by the approach of Nielsen and Shimotsu (2007) using $\bar{d}_*$, the average of the estimated integration orders of daily high and low prices from the ELW estimator with $m_d = T^{0.6}$ as bandwidth parameter, in the fractional cointegration analysis for both $v(T) = m_L^{-0.45}$ and $v(T) = m_L^{-0.05}$, with $m_L = T^{0.5}$.

				Rank estimates					
		Eigenvalues		$v(T) = m_L^{-0.45}$			$v(T) = m_L^{-0.05}$		
Stock	$\bar{d}_*$	$\hat{\delta}_1$	$\hat{\delta}_2$	$L(0)$	$L(1)$	$\hat{r}$	$L(0)$	$L(1)$	$\hat{r}$
ITAUB4	0.9024	0.2250	0.0006	-1.6855	-1.8373	1	-0.3716	-1.1803	1
BBDC4	0.9367	0.2089	0.0006	-1.6855	-1.8373	1	-0.3716	-1.1804	1
ABEV3	0.9954	0.1225	0.0005	-1.6831	-1.8330	1	-0.3702	-1.1766	1
PETR4	0.9900	0.6228	0.0021	-1.6783	-1.8324	1	-0.3675	-1.1770	1
VALE5	1.1019	0.2946	0.0006	-1.6807	-1.8362	1	-0.3689	-1.1803	1
BBAS3	0.9593	0.5299	0.0016	-1.6831	-1.8355	1	-0.3702	-1.1791	1
BRFS3	0.9969	0.1668	0.0005	-1.6831	-1.8353	1	-0.3702	-1.1789	1
UGPA3	0.9972	0.1371	0.0007	-1.6561	-1.8185	1	-0.3553	-1.1681	1
CIEL3	0.9981	0.1809	0.0008	-1.6855	-1.8336	1	-0.3716	-1.1766	1
KROT3	0.9721	0.4986	0.0024	-1.6394	-1.8101	1	-0.3466	-1.1638	1
VIVT4	0.9102	0.1810	0.0006	-1.6831	-1.8349	1	-0.3702	-1.1785	1
LREN3	0.9842	0.2164	0.0013	-1.6831	-1.8295	1	-0.3702	-1.1730	1
CCRO3	0.9122	0.2390	0.0012	-1.6831	-1.8314	1	-0.3702	-1.1750	1
RADL3	1.0569	0.1508	0.0008	-1.6831	-1.8308	1	-0.3702	-1.1744	1
JBSS3	0.9174	0.5854	0.0031	-1.6831	-1.8311	1	-0.3702	-1.1747	1
CPFE3	0.9862	0.1360	0.0005	-1.6855	-1.8350	1	-0.3716	-1.1780	1
HYPE3	0.9707	0.3071	0.0010	-1.6831	-1.8353	1	-0.3702	-1.1789	1
EMBR3	0.9873	0.2646	0.0012	-1.6831	-1.8327	1	-0.3702	-1.1762	1
WEGE3	0.9258	0.1452	0.0006	-1.6831	-1.8330	1	-0.3702	-1.1766	1
PCAR4	1.0687	0.2827	0.0156	-1.6783	-1.7248	1	-0.3675	-1.0694	1

The profile likelihood function is maximized both under the hypothesis H_r and under H_n considering the LR test statistic computed as follows:

$$LR(q) = 2\log\left(L(\hat{d}_n, \hat{b}_n, n)/L(\hat{d}_r, \hat{b}_r, r)\right), \qquad (11)$$

where $q = n - r$ and

$$L(\hat{d}_n, \hat{b}_n, n) = \max_{d,b} L(d,b,n), \text{ and } L(\hat{d}_r, \hat{b}_r, r) = \max_{d,b} L(d,b,r). \tag{12}$$

The asymptotic distribution of $LR(q)$ depends qualitatively (and quantitatively) on the parameter b. In the case of "weak integration", $0 < b < 0.5$, $LR(q)$ has a standard asymptotic distribution (NIELSEN & MORIN, 2016):

$$LR(q) \xrightarrow{D} \chi^2(q^2), \; 0 < b < 0.5. \tag{13}$$

Otherwise, in the case of "strong cointegration", when $0.5 < b \leq d$, asymptotic theory is nonstandard and

$$LR(q) \xrightarrow{D} \mathrm{Tr}\left\{ \int_0^1 \mathrm{d}W(s)F(s)' \left(\int_0^1 F(s)F(s)'\mathrm{d}s \right)^{-1} \int_0^1 F(s)\mathrm{d}W(s)' \right\}, \; b \geq 1/2, \tag{14}$$

where the vector process $\mathrm{d}W$ is the increment of ordinary (non-fractional) vector standard Brownian motion of dimension $q = p - r$ (NIELSEN & MORIN, 2016). The vector process F depends on the deterministics in a similar way as in the CVAR model in Johansen (1995). In the model with no determinist term $F(u) = W_b(u)$, otherwise, if the restricted constant term is included in the model, then $F(u) = \left(W_b'(u), 1\right)'$, where $W_b(u) = \Gamma(b)^{-1} \int_0^u (u-s)^{b-1} \mathrm{d}W(s)$ is vector fractional type-II Brownian motion.

Table 6 shows the results of the cointegration test of Johansen and Nielsen (2012). For all stocks, a significant cointegration relationship was found. For $r = 0$, larger values of the likelihood ratio (LR) statistics indicates the rejection the null hypothesis of zero cointegrating relationship. Otherwise, when $r = 1$, the LR statistics are smaller and the corresponding p-values indicate that we cannot reject the null of one cointegrating relationship.

3.2 Empirical FCVAR model

Based on the previous evidence of one significant cointegrating vector for the stocks traded at the BM&FBOVESPA, a fractionally cointegrating VAR (FCVAR) model was estimated for the daily high and low prices. In all cases, we set $p = 1$ for the short-term deviations, which is sufficient to capture the autocorrelation of the residuals. Also, as stated by MacKinnon and Nielsen (2014), a single lag is usually sufficient in the fractional model, in contrast with the standard cointegrated VAR where more lags are required to account for the serial correlation in the residuals. The FCVAR model was estimated for the case when $d \neq b$, since all estimates reported earlier rejects the hypothesis where d and b are close to equality (see Table 6).

Table 6. Likelihood ratio (*LR*) statistics and p-values from the cointegration test by Johansen and Nielsen (2012) for each rank $r = 0, 1, 2$, and the corresponding estimates of the parameter of the fractional order of integration ($\hat{d}$) and the parameter of the cointegration gap ($\hat{b}$).

	$r=0$				$r=1$				$r=2$	
Stock	$\hat{d}$	$\hat{b}$	*LR*	p-value	$\hat{d}$	$\hat{b}$	*LR*	p-value	$\hat{d}$	$\hat{b}$
ITAUB4	0.660	0.428	31.464	0.000	1.017	0.333	0.138	0.710	0.998	0.332
BBDC4	0.549	0.442	29.178	0.000	0.994	0.526	0.182	0.673	0.977	0.522
ABEV3	0.704	0.434	22.535	0.000	1.041	0.492	0.458	0.509	0.987	0.258
VALE5	0.722	0.435	17.597	0.001	0.949	0.200	0.186	0.666	0.937	0.153
PETR4	0.777	0.393	29.662	0.000	1.112	0.228	2.440	0.118	1.141	0.377
BBAS3	0.709	0.458	30.939	0.000	1.020	0.483	1.457	0.227	1.069	0.506
BRFS3	0.678	0.430	31.713	0.000	1.012	0.495	1.890	0.169	0.981	0.467
UGPA3	0.637	0.458	38.847	0.000	1.036	0.656	0.401	0.573	1.016	0.647
CIEL3	0.677	0.446	20.704	0.000	1.043	0.353	2.521	0.112	0.992	0.277
KROT3	0.583	0.449	18.863	0.001	1.021	0.354	3.423	0.064	0.945	0.171
VIVT4	0.685	0.444	29.783	0.000	0.462	0.462	3.253	0.079	0.991	0.602
LREN3	0.515	0.409	16.802	0.002	1.006	0.534	0.193	0.683	0.939	0.436
CCRO3	0.600	0.444	29.294	0.000	1.011	0.473	1.859	0.173	0.967	0.433
RADL3	0.572	0.463	25.499	0.000	1.069	0.513	1.762	0.145	1.013	0.440
JBSS3	0.557	0.438	57.328	0.000	0.987	0.714	0.136	0.725	0.999	0.717
CPFE3	0.936	0.300	52.309	0.000	1.116	0.341	0.786	0.375	1.144	0.371
HYPE3	0.709	0.470	40.780	0.000	0.968	0.397	1.508	0.219	0.922	0.380
EMBR3	0.623	0.460	33.862	0.000	1.025	0.705	0.137	0.721	1.022	0.703
WEGE3	0.735	0.445	24.928	0.000	0.835	0.010	0.129	0.762	1.034	0.627
PCAR4	0.457	0.350	27.375	0.000	1.009	0.706	0.135	0.715	1.009	0.906

Table 7 reports the FCVAR estimates for the high and low prices of the most traded stocks in the Brazilian stock market. The results are similar for all stocks. First, the parameters estimates of the fractional integration order and the cointegration gap, $\hat{d}$ and $\hat{b}$ respectively, are significantly different from zero and different from each other. Estimates of $\hat{d}$ indicate that daily high and low prices are integrated of an order close to the unity (except for BRFS3 and JBSS3 stocks that show the lower values of $\hat{d}$, approximately 0.84). The orders of integration of daily prices are smaller than unity in 16 out of 20 cases. Regarding the cointegrating vector, $\hat{\beta}$, the estimates are very close to the vector of $(1,-1)$. Since the range is defined as the difference between the high and low daily prices, i.e., $(p_t^H - p_t^L)$, it is expected the cointegrating vector to be $(1,-1)$. The results suggest that a linear combination of the daily high and low prices (the range) is integrated of a non-zero order, and the range is in the stationary region in most of the cases ($d-b<0.5$), with the exception of ranges of the PETR4, BRFS3 and CPFE3 stocks[11].

The estimates of the adjustment coefficients, $\hat{\alpha}^H$ and $\hat{\alpha}^L$, which describe the speed of adjustment of p_t^H and p_t^L toward equilibrium, are significantly different from zero (Table 7). In all cases, $\hat{\alpha}^H$ is negative and $\hat{\alpha}^L$ is positive, indicating that they move in opposite directions to restore equilibrium after a shock to the system occurs. Considering the absolute value of theses parameters estimates, in 50% of the cases, $\hat{\alpha}^H$ estimates are smaller than $\hat{\alpha}^L$, implying that the correction in the equation for daily lows overshoots the long-run equilibrium. These results were also verified by Baruník and Dvořáková (2015) and Caporin et al. (2013), however, in more than 50% of the cases $\hat{\alpha}^H$ estimates were smaller than $\hat{\alpha}^L$.

Concerning the short-run dynamics parameters estimates $\Gamma_1 = (\hat{\gamma}_{11},\ldots,\hat{\gamma}_{22})$, the coefficients of the lagged daily highs and lows are mostly positive, which suggests an indication of spill-over effects (Table 7)[12]. Finally, the residuals were also tested for the remaining autocorrelation and heteroskedasticity. In most cases, the null of no autocorrelation was rejected according to the Ljung-Box Q-test, but based on the visualization of the autocorrelation functions, the dependency is weak, and it disappears after the second lag. Some heteroskedasticity was also detected by the autocorrelation function of squared residuals, however, it is very weak[13].

[11] This finding differs from Baruník and Dvořáková (2015) and Caporin et al. (2013), where the ranges fall mostly in the non-stationary region.

[12] Cheung (2007) states that negative coefficients imply a regressive behavior, whereas positive coefficients are an indication of spill-over effects. In this case, higher daily highs tend to fall to a lower level, lower daily highs tend to drift up to a higher level, and higher daily lows lead to higher daily highs (BARUNÍK & DVOŘÁKOVÁ, 2015).

[13] These results are not reported here to avoid very exhaustive analysis.

Table 7. FCVAR model estimates results. Standard errors are shown below the parameters estimates in brackets.

Stock	$\hat{d}$	$\hat{b}$	$\hat{\beta}$	$\hat{\alpha}_H$	$\hat{\alpha}_L$	$\hat{\gamma}_{11}$	$\hat{\gamma}_{12}$	$\hat{\gamma}_{21}$	$\hat{\gamma}_{22}$
ITAUB4	0.913	0.535	(1,-1.008)	-0.490	0.914	0.174	0.189	0.135	0.326
	(0.029)	(0.067)		(0.171)	(0.243)	(0.145)	(0.140)	(0.161)	(0.187)
BBDC4	0.928	0.542	(1,-1.009)	-0.651	0.810	0.265	0.058	0.183	0.230
	(0.030)	(0.064)		(0.188)	(0.211)	(0.152)	(0.144)	(0.150)	(0.163)
ABEV3	1.002	0.703	(1,-1.008)	-0.636	0.538	0.022	0.132	0.020	0.201
	(0.028)	(0.057)		(0.107)	(0.090)	(0.074)	(0.077)	(0.064)	(0.079)
VALE5	0.854	0.388	(1,-1.010)	-1.754	0.592	0.446	0.115	-0.058	0.768
	(0.042)	(0.053)		(0.457)	(0.260)	(0.306)	(0.327)	(0.210)	(0.275)
PETR4	0.922	0.389	(1,-1.008)	-0.973	0.973	0.261	0.128	0.207	0.292
	(0.041)	(0.069)		(0.411)	(0.431)	(0.332)	(0.326)	(0.326)	(0.351)
BBAS3	0.935	0.631	(1,-1.010)	-0.204	0.721	-0.021	0.399	0.144	0.296
	(0.039)	(0.057)		(0.105)	(0.151)	(0.098)	(0.112)	(0.116)	(0.126)
BRFS3	0.828	0.322	(1,-1.013)	-0.883	1.872	0.536	0.226	-0.238	1.224
	(0.040)	(0.032)		(0.386)	(0.554)	(0.312)	(0.364)	(0.439)	(0.470)
UGPA3	1.004	0.725	(1,-1.006)	-0.554	0.546	0.126	0.112	-0.004	0.235
	(0.025)	(0.065)		(0.118)	(0.114)	(0.086)	(0.086)	(0.085)	(0.097)
CIEL3	0.955	0.516	(1,-1.010)	-1.002	0.743	0.369	-0.180	0.079	0.193
	(0.029)	(0.074)		(0.267)	(0.210)	(0.196)	(0.175)	(0.143)	(0.167)
KROT3	0.950	0.513	(1,-1.015)	-0.898	0.677	0.314	-0.035	0.233	0.146
	(0.048)	(0.089)		(0.306)	(0.278)	(0.233)	(0.214)	(0.211)	(0.220)
VIVT4	0.974	0.687	(1,-1.007)	-0.490	0.708	-0.025	0.239	-0.084	0.301
	(0.023)	(0.057)		(0.106)	(0.130)	(0.077)	(0.083)	(0.091)	(0.105)
LREN3	0.949	0.620	(1,-1.011)	-1.347	0.254	0.184	-0.115	-0.009	0.219
	(0.028)	(0.069)		(0.234)	(0.078)	(0.128)	(0.133)	(0.059)	(0.093)
CCRO3	0.943	0.565	(1,-1.011)	-0.528	0.915	0.090	0.151	-0.148	0.476
	(0.029)	(0.067)		(0.149)	(0.215)	(0.119)	(0.116)	(0.148)	(0.177)
RADL3	1.020	0.617	(1,-1.008)	-0.786	0.767	0.353	-0.140	0.082	0.130
	(0.032)	(0.063)		(0.152)	(0.138)	(0.128)	(0.104)	(0.094)	(0.101)
JBSS3	0.857	0.662	(1,-1.020)	-0.146	0.470	-0.002	0.443	0.271	0.224
	(0.041)	(0.061)		(0.067)	(0.110)	(0.067)	(0.083)	(0.089)	(0.092)
CPFE3	0.968	0.304	(1,-1.007)	-2.130	1.894	1.171	-0.845	-0.337	0.885
	(0.029)	(0.075)		(0.086)	(0.085)	(0.159)	(0.115)	(0.169)	(0.155)
HYPE3	0.936	0.530	(1,-1.009)	-0.655	0.865	0.302	0.143	0.148	0.414
	(0.036)	(0.057)		(0.162)	(0.198)	(0.130)	(0.132)	(0.138)	(0.168)
EMBR3	0.973	0.701	(1,-1.011)	-0.522	0.595	0.060	0.175	0.006	0.321
	(0.030)	(0.054)		(0.091)	(0.108)	(0.068)	(0.072)	(0.080)	(0.095)
WEGE3	0.967	0.617	(1,-1.011)	-0.617	0.698	0.273	-0.004	0.239	0.147
	(0.034)	(0.062)		(0.124)	(0.132)	(0.111)	(0.090)	(0.096)	(0.096)
PCAR4	1.011	0.729	(1,-1.006)	-1.104	0.017	0.038	-0.074	-0.006	0.091
	(0.028)	(0.052)		(0.085)	(0.010)	(0.040)	(0.118)	(0.007)	(0.045)

4. Predictability of daily high and low prices and trading performance

Besides the advantages of describing the dynamics of high and low asset prices and their difference, the range, the forecasting ability of the FCVAR modeling framework was also examined in the Brazilian stock market. Forecasts were performed using the FCVAR in an out-of-sample set comprised by the last three years of data. As competing models, we consider the VECM model of Cheung (2007); the random walk, RW; the ARIMA model; the 5-day moving average, MA_5; and the 22-day moving average, MA_{22}; the latter two of which correspond to weekly and monthly averages respectively and are very employed by technical analysts.

The Diebold and Mariano (1995) test is carried out to measure the forecasting superiority of the FCVAR, focusing on the mean squared error (MSE) of the forecasts. The error of the model i for the h-step ahead forecasting horizon is defined by:

$$\varepsilon^H_{t+h,i} = p^H_{t+h} - \hat{p}^H_{t+h,i}, \tag{15}$$

for the daily high, and

$$\varepsilon^L_{t+h,i} = p^L_{t+h} - \hat{p}^L_{t+h,i}, \tag{16}$$

for the daily low, with $i = \text{FCVAR}, \text{VECM}, \text{RW}, \text{ARIMA}, \text{MA}_5, \text{MA}_{22}$, where p^H_t (p^L_t) and $\hat{p}^H_t$ ($\hat{p}^L_t$) are the actual and predicted high (low) prices at t, respectively.

It is worth noting that not only one-step-ahead forecasting is conducted to assess the prediction performance of fractionally cointegration models for high and low asset prices, as made by Caporin (2013), but also five- and ten-step-ahead forecasting are performed to examine the medium- and long-term forecasting ability of the empirical FCVAR and selected competitors.

Table 8 shows summary results of the Diebold and Mariano (1995) test for the out-of-sample forecasts of daily high and low log-prices obtained using the FCVAR against the benchmark models[14].

[14] For the sake of brevity, detailed results are not presented here but are available upon request.

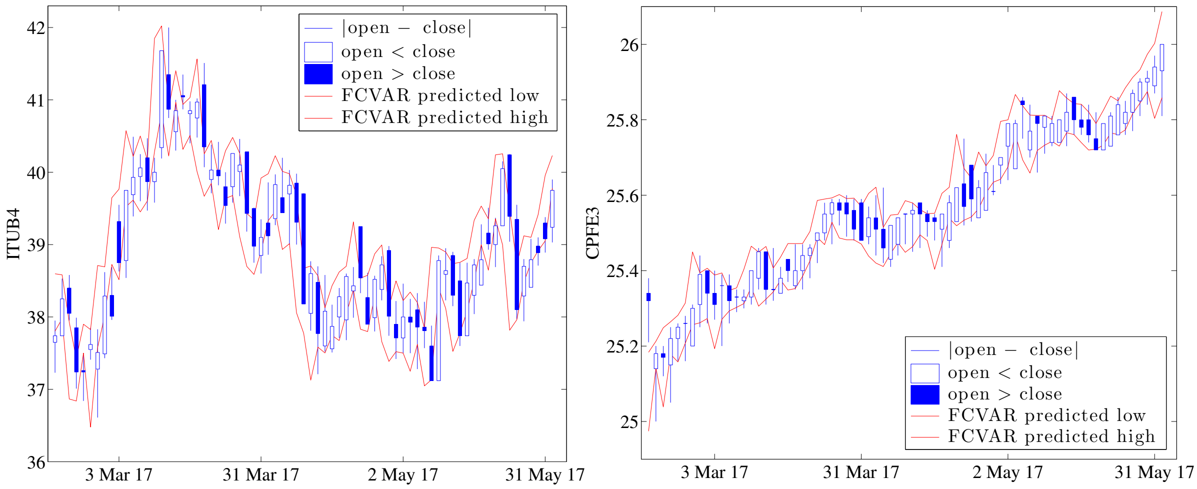
Table 8. Summary of Diebold-Mariano test. "$>$" indicates the number of cases in which the FCVAR forecasts over-perform with respect to the corresponding model at 95% confidence. "$=$" indicates the number of cases in which the performance of the FCVAR is statistically equal to that of the corresponding model, whereas "$<$" indicates under-performance of the FCVAR.

	VECM			RW			ARIMA			MA_5			MA_{22}		
Price	$>$	$=$	$<$	$>$	$=$	$<$	$>$	$=$	$<$	$>$	$=$	$<$	$>$	$=$	$<$
Panel A: one-step-ahead prediction horizon															
p^H_{t+1}	8	12	0	17	3	0	18	2	0	20	0	0	20	0	0
p^L_{t+1}	9	11	0	15	5	0	17	2	0	20	0	0	20	0	0
Panel B: five-step-ahead prediction horizon															
p^H_{t+5}	8	12	0	15	5	0	16	4	0	20	0	0	20	0	0
p^L_{t+5}	7	13	0	13	7	0	13	7	0	20	0	0	20	0	0
Panel C: ten-step-ahead prediction horizon															
p^H_{t+10}	6	14	0	13	7	0	17	3	0	20	0	0	20	0	0
p^L_{t+10}	5	15	0	11	9	0	15	5	0	20	0	0	20	0	0

From the experimental results obtained, the FCVAR approach in general consistently outperforms all of other competitors (Table 8). Overall, the rankings from best to worst are: FCVAR, VECM, ARIMA, RM, MA_5, MA_{22}. As far as the comparison between the FCVAR and VECM, the former almost wins in 50% of the cases. When it is not the case, the methods can be considered as equally accurate. As expected, the moving average methodologies performed worst. When comparing the performance of each method across the three prediction horizons (i.e., 1, 5, and 10), the superior performance of FCVAR over the remaining methods is still verified. However, predictions of FCVAR and VECM tend to be equally accurate with the increase in prediction horizon. Summing up, the results indicate the predictability of the daily high and low prices in the Brazilian stock markets. Moreover, the use of a long memory framework such as the FCVAR do improve forecasting performance in short- and long-term prediction horizons.

Figure 3 illustrates the performance of FCVAR modeling framework for daily high and low forecasting by showing the candlesticks plots of ITUB4 and CPFE3 stocks, based on the observed prices of the equities with the corresponding predicted high-low bands by FCVAR for the last three months of data, considering one-step-ahed predictions. It is interesting to note that

FCVAR provide a good fit of the high-low dispersion, indicating the potential of the proposed method which can enhances chart analysis, a tool often used by technical traders.



(a) ITUB4 candlestick and FCVAR predicted high−low bands

(b) CPFE3 candlestick and FCVAR predicted high−low bands

Figure 3. ITUB4 (a) and CPFE3 (b) candlesticks and FCVAR predicted high-low bands.

To evaluate the forecasts in a perspective of profit-seeking a trading strategy is performed as an illustrative example. Let p_t^O and p_t^C be the opening and closing stock prices at t, respectively, and $\hat{p}_{t+h}^H$ and $\hat{p}_{t+h}^L$ be the forecasted high and low prices for day $t+h$ after market closes on day t. The trading strategy is comprised by four steps (XIONG ET AL., 2017): i) on a given day t, a 'buy' signal for the asset is generated if $\hat{p}_{t+h}^H - p_t^O > p_t^O - \hat{p}_{t+h}^L$; ii) if the 'buy' signal is observed for k consecutive days beginning with day t, buy the asset on day $t+k-1$ using the closing value p_{t+k-1}^C; otherwise, hold the capital; iii) on another day s subsequent to buying the asset, a 'sell' signal is generated if $\hat{p}_{s+h}^H - p_s^O < p_s^O - \hat{p}_{s+h}^L$; iv) sell the asset on day $s+k-1$ using the closing value p_{s+k-1}^C of that day if a 'sell' signal has been observed for k consecutive trading days beginning with day s; otherwise, hold the asset.

Notice that the predicting horizon in this paper is one-, five- and ten-step-aheads ($h = 1,5,10$). The observed consecutive trading days k has to be set in advance and do not change as the steps of the trading strategy are conducted. In this work we set $k = 2$ as an example. A one-time 0.1% deduction was considered in order to mimic the transaction cost. Also, it is supposed that the investors can enter the market at any time during the evaluation period.

Table 9 shows the annualized returns from a trading strategy concerning high and low stock prices forecasting using different approaches, including the one suggested in this paper, the

FCVAR model. In addition, Table 10 presents the percentage os trades resulting in positive returns. Generally speaking, both the annualized returns and percentage of trades resulting in positive returns suggest that the FCVAR performs quite well, corresponding to higher values of both metrics. It is worthy to note that the percentage of profitable trades is always larger than 50% for FCVAR.

As far as the comparison among the forecasting methods in terms of the average annualized returns and percentage of trades with a positive annualized return is concerned, for all stocks FCVAR provided superior average results than the alternatives (Tables 9 and 10). The VECM method showed very similar results with FCVAR. RW showed the worst results. Further, notice that the values of annualized returns and percentage of trades with a positive annualized return of all methodologies are reduced but not significantly with the increasing of the forecasting horizon. Our findings are in line with the results of Caporin et al. (2013), which concern the US stock market, indicating that a fractionally cointegration approach is able to improve TA strategies based on high and low prices forecasting.

Table 9. Annualized returns of a simple trading strategy based on high and low prices forecasting using FCVAR and alternative benchmarks for different prediction horizons (h).

	$h=1$						$h=5$						$h=10$					
Stock	FCVAR	VECM	RW	ARIMA	MA_5	MA_{22}	FCVAR	VECM	RW	ARIMA	MA_5	MA_{22}	FCVAR	VECM	RW	ARIMA	MA_5	MA_{22}
ITAUB4	33.86%	28.34%	16.94%	24.76%	25.11%	19.79%	31.40%	29.46%	23.75%	26.71%	26.29%	26.25%	30.99%	25.78%	21.67%	19.86%	23.00%	22.98%
BBDC4	32.60%	29.46%	20.39%	22.51%	28.60%	24.07%	32.71%	29.29%	27.94%	28.55%	26.43%	26.41%	28.06%	29.76%	24.41%	21.63%	22.36%	21.05%
ABEV3	30.78%	25.34%	22.81%	25.60%	24.29%	22.71%	36.28%	32.42%	23.80%	25.97%	30.58%	24.21%	29.29%	27.21%	23.23%	22.56%	26.61%	21.60%
VALE5	33.06%	28.93%	24.67%	24.79%	24.51%	22.83%	35.24%	31.30%	25.24%	26.05%	26.19%	26.06%	27.54%	28.44%	21.45%	24.63%	24.42%	23.46%
PETR4	29.75%	27.82%	19.47%	28.67%	21.61%	19.95%	31.67%	30.31%	22.44%	22.66%	25.29%	28.89%	29.47%	25.66%	22.84%	24.54%	24.27%	25.81%
BBAS3	32.52%	26.23%	16.59%	25.63%	21.01%	26.16%	29.91%	27.56%	23.38%	25.08%	28.34%	31.44%	31.43%	25.26%	21.41%	22.05%	21.51%	23.36%
BRFS3	33.47%	28.12%	20.90%	25.83%	25.76%	25.17%	33.01%	27.27%	22.40%	24.03%	32.11%	23.88%	31.10%	27.04%	20.13%	23.67%	23.81%	19.46%
UGPA3	32.46%	24.22%	21.40%	23.51%	24.19%	23.64%	34.51%	26.66%	25.21%	25.74%	27.94%	33.08%	33.30%	26.75%	21.75%	21.55%	23.95%	23.06%
CIEL3	33.60%	23.73%	19.96%	20.69%	26.30%	26.86%	35.89%	34.89%	26.84%	27.29%	28.54%	30.87%	30.33%	29.35%	21.08%	23.22%	21.37%	21.91%
KROT3	35.57%	25.73%	17.74%	22.62%	22.39%	21.37%	34.28%	31.95%	23.43%	28.99%	25.92%	27.34%	27.08%	28.29%	20.23%	25.36%	22.76%	21.57%
VIVT4	33.67%	26.12%	25.36%	21.81%	30.23%	25.10%	31.97%	30.66%	25.29%	29.46%	31.41%	26.12%	32.47%	28.17%	24.72%	22.37%	22.47%	21.63%
LREN3	32.55%	25.74%	14.84%	22.22%	25.23%	24.91%	34.68%	28.45%	23.59%	26.20%	28.15%	25.08%	31.90%	24.60%	21.46%	21.59%	23.96%	20.17%
CCRO3	33.19%	24.58%	19.28%	24.57%	27.82%	24.91%	31.87%	28.14%	24.68%	25.86%	29.00%	29.07%	29.28%	25.46%	26.73%	21.25%	21.51%	21.40%
RADL3	32.82%	28.81%	17.33%	22.23%	24.25%	21.79%	32.27%	31.19%	26.65%	27.79%	25.89%	25.09%	30.41%	27.56%	20.38%	23.32%	24.86%	20.38%
JBSS3	29.86%	24.85%	18.97%	21.52%	24.43%	24.85%	32.86%	33.02%	24.72%	27.23%	29.68%	25.70%	31.19%	26.57%	22.19%	24.37%	25.98%	21.31%
CPFE3	31.39%	28.43%	21.26%	22.54%	26.82%	22.41%	31.76%	27.15%	22.63%	27.70%	28.71%	29.20%	30.64%	29.52%	24.32%	25.71%	22.63%	22.37%
HYPE3	34.18%	29.85%	20.83%	23.06%	26.45%	21.30%	33.59%	29.39%	22.96%	27.72%	25.46%	28.03%	31.07%	29.61%	26.73%	22.19%	23.11%	22.49%
EMBR3	32.03%	28.20%	18.79%	24.26%	21.75%	20.41%	32.39%	30.05%	24.19%	23.88%	27.97%	29.23%	29.94%	26.15%	21.64%	22.38%	23.93%	21.64%
WEGE3	33.09%	28.40%	22.36%	21.40%	20.60%	26.91%	33.56%	26.61%	24.15%	27.32%	26.94%	28.21%	30.05%	28.78%	19.34%	21.02%	25.05%	21.77%
PCAR4	30.39%	27.41%	21.99%	24.07%	21.82%	21.45%	33.22%	31.71%	22.94%	26.24%	28.12%	25.86%	30.22%	29.71%	22.69%	24.84%	23.96%	22.18%

Table 10. Percentage of trades with positive annualized returns of a simple trading strategy based on high and low prices forecasting using FCVAR and alternative benchmarks for different prediction horizons (h).

	$h = 1$						$h = 5$						$h = 10$					
Stock	FCVAR	VECM	RW	ARIMA	MA_5	MA_{22}	FCVAR	VECM	RW	ARIMA	MA_5	MA_{22}	FCVAR	VECM	RW	ARIMA	MA_5	MA_{22}
ITAUB4	55.55%	54.08%	51.22%	47.24%	54.86%	55.34%	62.53%	52.26%	50.68%	48.72%	52.27%	49.13%	55.54%	59.00%	57.28%	54.32%	50.48%	49.34%
BBDC4	59.05%	48.82%	50.85%	50.49%	51.71%	50.32%	56.49%	51.47%	49.13%	53.63%	49.73%	56.07%	60.34%	51.54%	56.49%	51.54%	51.90%	52.42%
ABEV3	48.00%	54.21%	52.36%	52.00%	51.44%	53.19%	58.10%	47.30%	48.36%	50.11%	53.88%	55.11%	58.94%	54.55%	55.88%	55.21%	53.48%	53.55%
VALE5	56.43%	56.76%	52.34%	53.59%	55.34%	51.32%	51.03%	51.40%	58.57%	51.50%	52.53%	51.30%	58.03%	55.42%	58.13%	55.93%	52.36%	55.13%
PETR4	54.96%	53.57%	49.16%	54.88%	48.64%	53.78%	52.74%	49.87%	56.14%	53.89%	52.17%	47.98%	58.54%	57.38%	57.32%	56.42%	53.90%	52.37%
BBAS3	50.57%	55.10%	51.42%	50.65%	52.84%	51.16%	53.31%	49.46%	52.36%	55.21%	52.97%	50.74%	54.34%	51.52%	55.60%	52.23%	54.16%	57.37%
BRFS3	52.93%	54.24%	51.05%	46.07%	53.23%	54.32%	57.29%	48.96%	47.98%	49.38%	52.44%	51.49%	57.90%	57.52%	53.27%	55.31%	50.70%	51.81%
UGPA3	55.03%	51.35%	53.19%	48.25%	50.21%	55.04%	53.29%	50.71%	49.08%	55.80%	49.91%	52.62%	61.95%	52.80%	53.37%	54.37%	52.66%	53.99%
CIEL3	63.76%	53.02%	54.45%	47.32%	51.49%	57.86%	56.13%	46.59%	51.01%	54.18%	51.55%	51.22%	58.34%	56.39%	57.98%	56.33%	47.58%	54.90%
KROT3	61.58%	50.00%	54.50%	57.22%	47.63%	52.34%	48.15%	54.90%	53.72%	52.22%	51.58%	53.05%	58.20%	58.03%	50.97%	55.55%	56.10%	53.34%
VIVT4	50.46%	54.69%	49.17%	48.61%	47.68%	46.70%	53.07%	53.66%	47.77%	51.87%	50.77%	52.92%	56.34%	57.12%	54.94%	56.52%	51.48%	55.49%
LREN3	62.29%	48.99%	51.71%	52.57%	51.18%	50.47%	51.71%	52.14%	44.98%	51.81%	55.42%	48.65%	58.03%	59.23%	50.13%	53.74%	49.99%	53.98%
CCRO3	56.06%	49.21%	48.22%	49.84%	52.91%	56.83%	49.53%	52.10%	47.44%	51.58%	50.05%	49.44%	58.66%	59.02%	57.55%	53.70%	52.46%	50.99%
RADL3	53.93%	49.93%	48.49%	52.98%	58.13%	49.79%	55.57%	49.97%	52.43%	52.46%	50.87%	49.97%	59.12%	54.38%	57.15%	56.82%	49.41%	48.24%
JBSS3	56.03%	43.96%	51.48%	48.18%	49.02%	55.69%	54.92%	55.05%	52.60%	52.54%	50.39%	50.58%	57.21%	56.92%	55.00%	48.37%	53.07%	58.13%
CPFE3	53.55%	56.23%	55.64%	46.33%	51.41%	53.26%	54.20%	51.83%	52.76%	54.63%	49.42%	51.07%	57.53%	53.56%	54.82%	52.74%	51.66%	51.48%
HYPE3	53.76%	53.11%	49.42%	46.28%	50.66%	57.07%	50.23%	50.20%	51.14%	56.52%	51.49%	51.93%	62.96%	52.50%	48.78%	50.67%	58.78%	53.39%
EMBR3	58.12%	50.09%	52.50%	51.82%	45.48%	47.21%	57.37%	55.98%	52.01%	53.66%	51.31%	44.02%	52.68%	58.79%	56.45%	53.07%	56.08%	54.63%
WEGE3	57.90%	56.04%	50.89%	49.89%	49.66%	52.33%	55.12%	51.57%	50.17%	51.83%	55.10%	50.71%	63.45%	56.20%	49.52%	55.85%	46.63%	53.42%
PCAR4	57.93%	47.41%	54.52%	49.83%	45.86%	49.40%	53.23%	50.55%	53.91%	54.09%	51.37%	55.13%	58.91%	56.05%	49.20%	56.33%	54.27%	50.67%

5. Conclusion

This work evaluated the predictability and dynamic properties of daily high and low stock prices in the Brazilian stock market. The motivation for examining maximum and minimum asset prices is that they provide valuable information regarding the dynamic process throughout a day, week, etc; they can be seen as references values for investors in order to place buy or sell orders; and are also related with the concept of volatility since their difference, the range, is highly efficient and robust estimator of variability. The modeling of daily high and low prices considered a fractionally cointegrated VAR model (FCVAR), which accounts for two fundamental patterns of these prices: their cointegrating relationship and the long-memory of their difference (i.e., the range), as the error correction term is allowed to fall into a non-stationary region. Additionally, this work also evaluated if high and low prices forecasts by FCVAR can improve technical analysis through a simple trading strategy.

The empirical analysis examined daily high and low prices of the twenty most traded stocks in the Brazilian stock exchange, the BM&BOVESPA, during the period from January 2010 to May 2017. The findings indicated that daily high and low prices are integrated of an order close to the unity, and the range displays long memory and is in the stationary region in most of the cases. For all stocks, a significant cointegration relationship was found between daily high and low prices. The empirical FCVAR model shows that high and low prices move in opposite directions to restore equilibrium after a shock to the system occurs. Also, the results evidence the predictability of daily highs and lows in the Brazilian stock market for different forecasting horizons, in which the fractionally approach conducts to better predictions than competitive methods and can improve trading strategies.

Future work shall include the estimation of the FCVAR with the restriction on the cointegrating vector β to be $(1,-1)$, which allows the interpretation of the difference $(d-b)$ as the order of integration of the range. The evaluation of the forecasts in terms of more sophisticated trading strategies is aso demanding and compelling, mainly considering equity intradaily trading. Further, the evidence of long memory in the range dynamics can lead to the development of volatility methods based on the forecasted range to improve derivatives pricing and risk analysis.

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